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Self-Duality for Multi-State Probabilistic Cellular Automata with Finite Range Interactions

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In this paper we study a criterion of self-duality for multi-state probabilistic cellular automata with finite range interactions and give some models which satisfy this criterion.

KEY WORDS: Duality; cellular automata; multi-state; finite range.

1. INTRODUCTION

The present paper treats necessary and sufficient conditions on self-duality for multi-state probabilistic cellular automata with finite range interactions on the integer lattice **Z**. Self-duality is a very useful technique in the study of probabilistic cellular automata. Because problems in uncountable state space (typically configurations of zeros and ones live in **Z**) can be reformulated as problems in countable state space (typically finite subsets of **Z**). For some applications of self-duality on probabilistic cellular automata (in particular, oriented bond percolation model) and the contact process, see Chaps. 4 and 5 of ref. 1. To describe the dynamics of probabilistic cellular automata, we introduce an interaction neighborhood $\mathcal{N} = \{-L, -(L-1), \ldots, L-1, L\}$ and a transition function

$$f : \{0, 1, \ldots, M-1\}^{\mathcal{N}} \times \{0, 1, \ldots, M-1\} \to [0, 1]$$

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The evolution of probabilistic cellular automata $\eta_n = \{\eta_n(x) \in \{0, 1,..., M-1\} : x \in \mathbf{Z}\}$ is given by

$$P(\eta_{n+1}(x) = m)$$
$$= f(\eta_n(x-L), \eta_n(x-(L-1)),..., \eta_n(x+(L-1)), \eta_n(x+L); m)$$

for any $n \in \{0, 1, 2,...\}$, $m \in \{0, 1,..., M-1\}$ and $x \in \mathbf{Z}$. Let $N = 2L+1$. We call the above model *M-state N-neighbor probabilistic cellular automata* in this paper. Remark that

$$\sum_{m=0}^{M-1} f(i_1 i_2 ... i_N; m) = 1$$

for any $(i_1, i_2,..., i_N) \in \{0, 1,..., M-1\}^N$. Furthermore, when we emphasize the initial states $A_i = \{x : \eta_0(x) = i\}$ $(i = 1, 2,..., M-1)$, we write $\eta_n^{A_1, A_2,..., A_{M-1}} \in \{0, 1,..., M-1\}^{\mathbf{Z}}$.

Let $\xi_{(m), n}^{A_1, A_2,..., A_{M-1}}$ denote the set of m's for the above mentioned M-state N-neighbor probabilistic cellular automata at time n starting from a set of m's $= A_m \subset \mathbf{Z}$. That is, $\xi_{(m), n}^{A_1, A_2,..., A_{M-1}} = \{x \in \mathbf{Z} : \eta_n^{A_1, A_2,..., A_{M-1}}(x) = m\}$ for each $m \in \{1, 2,..., M-1\}$. We also call this set-valued process M-state N-neighbor probabilistic cellular automata. We sometimes write it ξ_n for short.

The M-state N-neighbor probabilistic cellular automata ξ_n is said to be *self-dual* with *self-duality parameters* x_m $(m = 1, 2,..., M-1)$ if

$$E\left(\prod_{m=1}^{M-1} x_m^{|\xi_{(m), n}^{A_1, A_2,..., A_{M-1}} \cap B_m|}\right) = E\left(\prod_{m=1}^{M-1} x_m^{|\xi_{(m), n}^{B_1, B_2,..., B_{M-1}} \cap A_m|}\right) \qquad (n = 0, 1,...)$$

for any $A_m, B_m \subset \mathbf{Z}$ with $|A_m| < \infty$ or $|B_m| < \infty$ $(m = 1, 2,..., M-1)$. The above equation is called *self-duality eqaution.*

In this setting, we obtain the following main result.

Theorem 1. We suppose that

$$f(00 \cdots 00; 0) = 1 \tag{1.1}$$

$$f(i_1 i_2 \cdots i_{N-1} i_N; m) = f(i_N i_{N-1} \cdots i_2 i_1; m) \tag{1.2}$$

for any $(i_1, i_2,..., i_N, m) \in \{0, 1,..., M-1\}^{N+1}$. Let

$$q(i_1 i_2 \cdots i_N; m; x_m) = x_m f(i_1 i_2 \cdots i_N; m) + 1 - f(i_1 i_2 \cdots i_N; m)$$

Then necessary and sufficient conditions on self-duality with self-duality parameters x_m $(m=1, 2,..., M-1)$ for M-state N-neighbor probabilisitic cellular automata with transition probabilities $f(i_1 i_2 \cdots i_N; m)$ are given by

$$q(i_1 i_2 \cdots i_N; m; x_m) = \prod_{k:i_k \neq 0} q(\overbrace{0 \cdots 0}^{k-1} m \overbrace{0 \cdots 0}^{N-k}; i_k; x_{i_k}) \tag{1.3}$$

for any $(i_1, i_2,..., i_N) \in \{0, 1,..., M-1\}^N$ except $(0, 0,..., 0)$ and $m \in \{1,..., M-1\}$, where $\prod_{k:i_k \neq 0}$ means $\prod_{k=1,2,...,N}$ with $i_k \neq 0$ for $(i_1, i_2,..., i_N)$.

We should note that Eq. (1.1) gives

$$q(00 \cdots 0; m; x_m) = 1 \qquad (m \in \{1,..., M-1\}) \tag{1.4}$$

Moreover in a special case of Eq. (1.3), we have

$$q(\overbrace{0 \cdots 0}^{k-1} u \overbrace{0 \cdots 0}^{N-k}; v; x_v) = q(\overbrace{0 \cdots 0}^{k-1} v \overbrace{0 \cdots 0}^{N-k}; u; x_u) \tag{1.5}$$

for any $u, v \in \{1,..., M-1\}$. Remark that $x_m = 1$ for any m is a trivial solution for Eq. (1.3) and noninformative.

Here we review briefly the results corresponding to Theorem 1. Katori *et al.*[2] gave the result on the Domany–Kinzel model (a special case of $M=2$ and $N=3$). See Example 1 in the next section.

Concerning recent studies on duality for continuous-time interacting particle systems, see refs. 3–7, for examples.

The rest of the paper is organized as follows. In Section 2, we will give some examples which satisfy Theorem 1. Section 3 is devoted to the proof of Theorem 1.

2. EXAMPLES

In this section we give three examples.

Example 1. If we consider $M=2$ and $N=3$ case with

$$f(000; 1) = f(010; 1) = 0$$

$$f(001; 1) = f(011; 1) = f(100; 1) = f(110; 1) = p_1$$

$$f(101; 1) = f(111; 1) = p_2$$

then our model is equivalent to the Domany–Kinzel model with two parameters p_1 and p_2 which was introduced by refs. 8 and 9. As special cases the Domany–Kinzel model is equivalent to the oriented bond percolation model ($p_1 = p$, $p_2 = 2p - p^2$) and the oriented site percolation model ($p_1 = p_2 = p$) on a square lattice. The two-dimensional mixed site-bond oriented percolation model with probabilities p_s of a site being open and p_b of a bond being open corresponds to the case of $p_1 = p_s p_b$ and $p_2 = p_s[1-(1-p_b)^2]$. The model with $(p_1, p_2) = (1, 0)$ becomes Wolfram's[10, 11] rule 90. For more detailed information, see pp. 90–98 in ref. 1. Then Eq. (1.3) with $x = x_m (m = 1,\dots, M-1)$ becomes the following one equation:

$$xp_2 + 1 - p_2 = (xp_1 + 1 - p_1)^2$$

The above result was obtained by ref. 2.

Example 2. We consider $M = 3$ and $N = 3$ case. Then Eq. (1.3) becomes

$$q(001; 2; x_2) = q(002; 1; x_1)$$
$$q(010; 2; x_2) = q(020; 1; x_1)$$
$$q(011; 1; x_1) = q(010; 1; x_1)\, q(001; 1; x_1)$$
$$q(011; 2; x_2) = q(020; 1; x_1)\, q(002; 1; x_1)$$
$$q(012; 1; x_1) = q(010; 1; x_1)\, q(001; 2; x_2)$$
$$q(012; 2; x_2) = q(020; 1; x_1)\, q(002; 2; x_2)$$
$$q(021; 1; x_1) = q(010; 2; x_2)\, q(001; 1; x_1)$$
$$q(021; 2; x_2) = q(020; 2; x_2)\, q(002; 1; x_1)$$
$$q(022; 1; x_1) = q(010; 2; x_2)\, q(001; 2; x_2)$$
$$q(022; 2; x_2) = q(020; 2; x_2)\, q(002; 2; x_2)$$
$$q(101; 1; x_1) = q(001; 1; x_1)^2$$
$$q(101; 2; x_2) = q(002; 1; x_1)^2$$
$$q(102; 1; x_1) = q(001; 1; x_1)\, q(001; 2; x_2)$$
$$q(102; 2; x_2) = q(002; 1; x_1)\, q(002; 2; x_2)$$
$$q(202; 1; x_1) = q(001; 2; x_2)^2$$
$$q(202; 2; x_2) = q(002; 2; x_2)^2$$

$$q(111; 1; x_1) = q(001; 1; x_1)^2 \, q(010; 1; x_1)$$

$$q(111; 2; x_2) = q(002; 1; x_1)^2 \, q(020; 1; x_1)$$

$$q(112; 1; x_1) = q(001; 1; x_1) \, q(010; 1; x_1) \, q(001; 2; x_2)$$

$$q(112; 2; x_2) = q(002; 1; x_1) \, q(020; 1; x_1) \, q(002; 2; x_2)$$

$$q(121; 1; x_1) = q(001; 1; x_1)^2 \, q(010; 2; x_2)$$

$$q(121; 2; x_2) = q(002; 1; x_1)^2 \, q(020; 2; x_2)$$

$$q(122; 1; x_1) = q(001; 1; x_1) \, q(010; 2; x_2) \, q(001; 2; x_2)$$

$$q(122; 2; x_2) = q(002; 1; x_1) \, q(020; 2; x_2) \, q(002; 2; x_2)$$

$$q(212; 1; x_1) = q(001; 2; x_2)^2 \, q(010; 1; x_1)$$

$$q(212; 2; x_2) = q(002; 2; x_2)^2 \, q(020; 1; x_1)$$

$$q(222; 1; x_1) = q(001; 2; x_2)^2 \, q(010; 2; x_2)$$

$$q(222; 2; x_2) = q(002; 2; x_2)^2 \, q(020; 2; x_2)$$

Example 3. We consider the following general M-state N-neighbor case: for any $(i_1, i_2,..., i_N) \in \{0, 1,..., M-1\}^N$ except $(0, 0,..., 0)$ and $m \in \{1,..., M-1\}$, transition probabilities are given by

$$f(i_1 i_2 \cdots i_N; m) = s_m \left[1 - \prod_{k \,:\, i_k \neq 0} (1 - p(i_k; m)) \right]$$

and $f(00 \cdots 00; 0) = 1$ with

$$p(i; m) = p(m; i) \qquad (i, m \in \{1,..., M-1\})$$

where $s_m \in \mathbf{R} \setminus \{0\}$ and $p(i_k; m) \in \mathbf{R}$ satisfy $f(i_1 i_2 \cdots i_N; m) \in [0, 1]$ for any $(i_1, i_2,..., i_N, m) \in \{0, 1,..., M-1\}^{N+1}$ and $\mathbf{R}$ be the set of real numbers. When

$$x_m = \frac{s_m - 1}{s_m} \qquad (m \in \{1,..., M-1\})$$

the above $f(i_1 i_2 ... i_N; m)$ satisfies Eq. (1.3) as follows: for any $(i_1, i_2,..., i_N) \in \{0, 1,..., M-1\}^N$ except $(0, 0,..., 0)$ and $m \in \{1,..., M-1\}$, we see that

$$q(i_1 i_2 \cdots i_N; m; x_m) = (x_m - 1)\, f(i_1 i_2 \cdots i_N; m) + 1$$
$$= (x_m - 1)\, s_m \left[1 - \prod_{k : i_k \neq 0} (1 - p(i_k; m)) \right] + 1$$
$$= \prod_{k : i_k \neq 0} [1 - p(i_k; m)]$$

on the other hand,

$$\prod_{k : i_k \neq 0} q(\overbrace{0 \cdots 0}^{k-1}\ m\ \overbrace{0 \cdots 0}^{N-k}; i_k; x_{i_k}) = \prod_{k : i_k \neq 0} [(x_{i_k} - 1)\, f(\overbrace{0 \cdots 0}^{k-1}\ m\ \overbrace{0 \cdots 0}^{N-k}; i_k) + 1]$$
$$= \prod_{k : i_k \neq 0} [(x_{i_k} - 1)\, s_{i_k}\, p(m; i_k) + 1]$$
$$= \prod_{k : i_k \neq 0} [1 - p(m; i_k)]$$
$$= \prod_{k : i_k \neq 0} [1 - p(i_k; m)]$$

So this is a typical class of models which satisfy Eq. (1.3).

3. PROOF OF THEOREM 1

In this section we prove Theorem 1. Let $\mathbf{a} = (a_j : j \in \mathbf{Z})$ and $\mathbf{b} = (b_j : j \in \mathbf{Z})$ with $a_j, b_j \in \{0, 1,..., M-1\}$ for any j, where, if $j \in A_m$ (resp. $\notin \bigcup_{l=1}^{M-1} A_l$), then $a_j = m$ (resp. $= 0$) and if $j \in B_m$ (resp. $\notin \bigcup_{l=1}^{M-1} B_l$), then $b_j = m$ (resp. $= 0$), for $m = 1, 2,..., M-1$. Furthermore we introduce the following notation: for any $(u_1, u_2,..., u_N, v) \in \{*, 0, 1,..., M-1\}^{N+1}$,

$${}_a\langle u_1 u_2 \cdots u_N \mid v \rangle_b = \sum_{j \in \mathbf{Z}} I_{u_1}(a_{j-L})\, I_{u_2}(a_{j-(L-1)}) \cdots I_{u_N}(a_{j+L})\, I_v(b_j)$$
$${}_b\langle u_1 u_2 \cdots u_N \mid v \rangle_a = \sum_{j \in \mathbf{Z}} I_{u_1}(b_{j-L})\, I_{u_2}(b_{j-(L-1)}) \cdots I_{u_N}(b_{j+L})\, I_v(a_j)$$

where $I_*(y) = 1$ for any y and, when $x \neq *$, $I_x(y) = 1$ (resp. $=0$), if $y = x$ (resp. $y \neq x$). Remark that the definition gives

$${}_a\langle u_1 \cdots u_{k-1} * u_{k+1} \cdots u_N \mid v \rangle_b = \sum_{u_k = 0}^{M-1} {}_a\langle u_1 \cdots u_{k-1} u_k u_{k+1} \cdots u_N \mid v \rangle_b \tag{3.1}$$
$${}_b\langle u_1 \cdots u_{k-1} * u_{k+1} \cdots u_N \mid v \rangle_a = \sum_{u_k = 0}^{M-1} {}_b\langle u_1 \cdots u_{k-1} u_k u_{k+1} \cdots u_N \mid v \rangle_a \tag{3.2}$$

for any $u_1,..., u_{k-1}, u_{k+1},..., u_N, v \in \{0, 1,..., M-1\}$ and $k \in \{1,..., N\}$. Moreover note that for any $u, v \in \{1,..., M-1\}$ and $k \in \{1,..., N\}$,

$$_a\langle \overbrace{*\cdots*}^{k-1}\ u\ \overbrace{*\cdots*}^{N-k} \mid v\rangle_b = {}_b\langle \overbrace{*\cdots*}^{N-k}\ v\ \overbrace{*\cdots*}^{k-1} \mid u\rangle_a \tag{3.3}$$

From the Markov property of ξ_n, it is sufficient to prove self-duality equation with time $n=1$, that is,

$$E\left(\prod_{m=1}^{M-1} x_m^{|\xi^{A_1, A_2,..., A_{M-1}}_{(m), 1} \cap B_m|}\right) = E\left(\prod_{m=1}^{M-1} x_m^{|\xi^{B_1, B_2,..., B_{M-1}}_{(m), 1} \cap A_m|}\right) \tag{3.4}$$

In our setting, we have

$$\begin{aligned}
\text{L.H.S. of Eq. (3.4)} &= \prod_{m=1}^{M-1} \prod_{k=1}^{N} \prod_{i_k=0}^{M-1} q(i_1 i_2 \cdots i_N; m; x_m)^{{}_a\langle i_1 i_2 \cdots i_N \mid m\rangle_b} \\
&= \prod_{m=1}^{M-1} \prod_{k=1}^{N} \prod_{i_k=0}^{M-1} \prod_{k: i_k \neq 0} q(\overbrace{0\cdots0}^{k-1}\ m\ \overbrace{0\cdots0}^{N-k}; i_k; x_{i_k})^{{}_a\langle i_1 i_2 \cdots i_N \mid m\rangle_b} \\
&= \prod_{m=1}^{M-1} \prod_{k=1}^{N} \prod_{i_k=0}^{M-1} \prod_{k: i_k \neq 0} q(\overbrace{0\cdots0}^{k-1}\ i_k\ \overbrace{0\cdots0}^{N-k}; m; x_m)^{{}_a\langle i_1 i_2 \cdots i_N \mid m\rangle_b} \\
&= \prod_{m=1}^{M-1} \prod_{k=1}^{N} \prod_{i_k=1}^{M-1} q(\overbrace{0\cdots0}^{k-1}\ i_k\ \overbrace{0\cdots0}^{N-k}; m; x_m)^{{}_a\langle \overbrace{*\cdots*}^{k-1}\ i_k\ \overbrace{*\cdots*}^{N-k} \mid m\rangle_b}
\end{aligned}$$

The second equality comes from Eqs. (1.3) and (1.4). We use Eq. (1.5) to get the third equality. The last equality is obtained by a standard inclusion-exclusion argument noting Eq. (3.1).

On the other hand, we obtain

$$\begin{aligned}
\text{R.H.S. of Eq. (3.4)} &= \prod_{m=1}^{M-1} \prod_{k=1}^{N} \prod_{i_k=1}^{M-1} q(\overbrace{0\cdots0}^{k-1}\ i_k\ \overbrace{0\cdots0}^{N-k}; m; x_m)^{{}_b\langle \overbrace{*\cdots*}^{k-1}\ i_k\ \overbrace{*\cdots*}^{N-k} \mid m\rangle_a} \\
&= \prod_{m=1}^{M-1} \prod_{k=1}^{N} \prod_{i_k=1}^{M-1} q(\overbrace{0\cdots0}^{k-1}\ i_k\ \overbrace{0\cdots0}^{N-k}; m; x_m)^{{}_a\langle \overbrace{*\cdots*}^{N-k}\ i_k\ \overbrace{*\cdots*}^{k-1} \mid i_k\rangle_b} \\
&= \prod_{m=1}^{M-1} \prod_{k=1}^{N} \prod_{i_k=1}^{M-1} q(\overbrace{0\cdots0}^{N-k}\ i_k\ \overbrace{0\cdots0}^{k-1}; m; x_m)^{{}_a\langle \overbrace{*\cdots*}^{N-k}\ i_k\ \overbrace{*\cdots*}^{k-1} \mid i_k\rangle_b} \\
&= \prod_{m=1}^{M-1} \prod_{k=1}^{N} \prod_{i_k=1}^{M-1} q(\overbrace{0\cdots0}^{N-k}\ m\ \overbrace{0\cdots0}^{k-1}; i_k; x_{i_k})^{{}_a\langle \overbrace{*\cdots*}^{N-k}\ i_k\ \overbrace{*\cdots*}^{k-1} \mid i_k\rangle_b} \\
&= \prod_{m=1}^{M-1} \prod_{j=1}^{N} \prod_{i_j=1}^{M-1} q(\overbrace{0\cdots0}^{j-1}\ m\ \overbrace{0\cdots0}^{N-j}; i_j; x_{i_j})^{{}_a\langle \overbrace{*\cdots*}^{j-1}\ i_k\ \overbrace{*\cdots*}^{N-j} \mid i_j\rangle_b}
\end{aligned}$$

The first equality is given by a similar computation as in the case of L.H.S of Eq. (3.4) noting Eq. (3.2). The second equality comes from Eq. (3.3). By using Eq. (1.2), we have the third equality. The fourth equality can be derived from Eq. (1.5).

Therefore we obtain the desired conclusion.

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